Combinability of Travelling Wave Solutions to Nonlinear Evolution Equation

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First, applying the Jacobi elliptic sine function expansion, basic travelling wave solutions of some nonlinear evolution equations are obtained. Second, according to the formal invariance of nonlinear evolution equations, the sister travelling wave solutions, which have the same wave speed as the basic ones, are derived. Finally, we show that a suitable linear combination of these two solutions results in a combined travelling wave solution, whose wave speed is different from those of the basic and sister travelling wave solutions. The combination of travelling wave solutions to nonlinear evolution equations must satisfy certain conditions. – PACS: 03.65.Ge.

Key words: Basic Travelling Wave Solution; Sister Travelling Wave Solution; Combined Travelling Wave Solution.

1. Introduction

It is important to find exact solutions of nonlinear wave equations in nonlinear problems. Recently, various methods have been proposed, such as the homogeneous balance method [1, 2], the hyperbolic function expansion method [3, 4], the sine-cosine method [5], the nonlinear transformation method [6 – 8], the trial function method [9, 10], and the Jacobi elliptic function expansion method [11, 12]. These methods can be applied to obtain travelling wave solutions which include shock or solitary wave solutions, periodic solutions [13 – 15] and so on.

In this paper, the basic travelling wave solutions to some nonlinear evolution equations are obtained by applying the Jacobi elliptic sine function expansion method [11, 12]. Then, according to the formal invariance of nonlinear evolution equations, the related travelling wave solutions, which are called sister travelling wave solutions of the basic ones, are derived. Finally, we show that a linear combination of these two solutions under certain conditions, where the wave speed is different from those of both the basic and sister travelling wave solution, are also a travelling wave solution to the nonlinear wave equation. This phenomenon is called the combinability of travelling wave solutions in this paper.

2. The First Kind of Nonlinear Evolution Equations

2.1. The Travelling Wave Solution

The first kind of nonlinear evolution equations include the Korteweg de Vries (KdV) equation

$$u_t + uu_x + \beta u_{xxx} = 0, \tag{1}$$

the Benjamin-Bona-Mahony (BBM) equation

$$u_t + c_0 u_x + uu_x + \beta u_{xxt} = 0, \tag{2}$$

the Boussinesq equation

$$u_{tt} - c_0^2 u_{xx} - \alpha u_{xxxx} - \beta (u^2)_{xx} = 0, \tag{3}$$

and the nonlinear Klein-Gordon equation $(\mathbf{A})$

$$u_{tt} - c_0^2 u_{xx} + \alpha u - \beta u^2 = 0. \tag{4}$$

We seek travelling wave solutions of (1) through (4) in the form

$$u = u(\xi), \quad \xi = k(x - ct), \tag{5}$$

where k and c are the wave number and wave speed, respectively.

Substituting (5) into (1) to (4), or integrating once or twice and taking the integration constants as zero, we obtain

$$e_1 \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + e_2 u + e_3 u^2 = 0. \tag{6}$$

Here e_1, e_2 and e_3 are constants which depend on the corresponding Eq. (1) to (4). For the KdV equation one finds

$$e_1 = \beta k^2, \quad e_2 = -c, \quad e_3 = \frac{1}{2}. \tag{7}$$

In case of the BBM equation one has

$$e_1 = \beta k^2 c, \quad e_2 = c - c_0, \quad e_3 = -\frac{1}{2}. \tag{8}$$

The Boussinesq equation leads to

$$e_1 = \alpha k^2, \quad e_2 = -(c^2 - c_0^2), \quad e_3 = \beta, \tag{9}$$

and the nonlinear Klein-Gordon equation $(\mathbf{A})$ to

$$e_1 = k^2 (c^2 - c_0^2), \quad e_2 = \alpha, \quad e_3 = -\beta. \tag{10}$$

2.2. The Basic Travelling Wave Solution

By using the Jacobi elliptic function expansion method [11, 12], $u(\xi)$ in (6) can be expressed by

$$u_1 = a_0 + a_2 \mathrm{sn}^2 \xi, \tag{11}$$

with

$$a_0 = -\frac{e_2}{2e_3} + \frac{2(1+m^2)e_1}{e_3}, \quad a_2 = -\frac{6e_1}{e_3} m^2, \tag{12}$$

and the modulus $m (0 < m < 1)$ satisfying

$$e_2^2 = 16(1 - m^2 + m^4) e_1^2, \tag{13}$$

where $\mathrm{sn}\xi$ is Jacobi elliptic sine function [16 – 19].

The constraint (13) can be used to determine the wave speed of the travelling wave solution u_1. For the KdV equation, it has the value

$$c_1^2 = 16(1 - m^2 + m^4) \beta^2 k^4. \tag{14}$$

For the BBM equation one has

$$(c_1 - c_0)^2 = 16(1 - m^2 + m^4) \beta^2 k^4 c_1^2, \tag{15}$$

for the Boussinesq equation

$$(c_1^2 - c_0^2)^2 = 16(1 - m^2 + m^4) \alpha^2 k^4, \tag{16}$$

and for the nonlinear Klein-Gordon equation $(\mathbf{A})$

$$\alpha^2 = 16(1 - m^2 + m^4) k^4 (c_1^2 - c_0^2)^2. \tag{17}$$

When $m \to 1$, the basic travelling wave solution (11) reduces to

$$u_1 = a_0 + a_2 \tanh^2 \xi, \quad \text{with } a_0 = -\frac{e_2}{2e_3} + \frac{4e_1}{e_3}, \quad a_2 = -\frac{6e_1}{e_3}. \tag{18}$$

2.3. The Sister Travelling Wave Solution

Next, we will show that there exists a sister travelling wave solution to (6), which is closely related to the just given basic solution. It takes the form

$$u_2 = a_0 + \frac{a_2}{m^2} \mathrm{ns}^2 \xi, \quad \text{with} \quad \mathrm{ns}\xi \equiv \frac{1}{\mathrm{sn}\xi}, \tag{19}$$

where a_0 and a_2 are the same constants as in (12) and (13). Denoting the wave speed of the travelling wave solution (19) as c_2, we have

$$c_2 = c_1, \tag{20}$$

i. e. the basic travelling wave and its related sister have the same speed, although their shapes are different.

The derivations of (19) and (20) are as follows. Setting

$$v = a_2 \mathrm{sn}^2 \xi, \quad u_1 = a_0 + v, \tag{21}$$

in (11), we can easily prove that v satisfies

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} = 2a_2 - 4(1+m^2)v + \frac{6m^2}{a_2} v^2. \tag{22}$$

We define

$$w = \frac{a_2}{m^2} \mathrm{ns}^2 \xi, \quad u_2 = a_0 + w. \tag{23}$$

Using $\mathrm{ns}\xi = 1/\mathrm{sn}\xi$, one can prove that w satisfies

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2} = 2a_2 - 4(1+m^2)w + \frac{6m^2}{a_2} w^2, \tag{24}$$

which has the same form as (22).

If u_1 satisfies (6), then u_2 also satisfies (6). This is shown by differentiating $u_2 = a_0 + w$ twice with respect to ξ and utilizing (12) and (24), which leads to

$$e_1 \frac{d^2 u_2}{d\xi^2} + e_2 u_2 + e_3 u_2^2 + \frac{e_2^2 - 16(1 - m^2 + m^4)e_1^2}{4e_3} = 0. \tag{25}$$

It is obvious that, if and only if (13) is satisfied, then (19) is another travelling wave solution of (6). Since the wave speed of (19) is the same as that of (11), we call (19) the sister travelling wave solution of the basic one (11).

When $m \to 1$, the sister travelling wave solution (19) reduces to

$$u_2 = a_0 + a_2 \coth^2 \xi, \quad \text{with } a_0 = -\frac{e_2}{2e_3} + \frac{4e_1}{e_3}, \quad a_2 = -\frac{6e_1}{e_3}. \tag{26}$$

2.4. *Combinability of Basic and Sister Travelling Wave Solutions*

We now show that a particular combination of $u_1 = a_0 + v$ and $u_2 = a_0 + w$, defined by

$$u_3 = a_0 + v + w = a_0 + a_2 \text{sn}^2 \xi + \frac{a_2}{m^2} \text{ns}^2 \xi, \tag{27}$$

is also a travelling wave solution of (6) with the constraint

$$e_2^2 = 16(1 + 14m^2 + m^4)e_1^2. \tag{28}$$

This constraint implies that the wave speed c_3 of (27) is different from c_1 of (11) and c_2 of (19). The wave velocity of the KdV equation is given by

$$c_3^2 = 16(1 + 14m^2 + m^4)\beta^2 k^4, \tag{29}$$

that of the BBM equation by

$$(c_3 - c_0)^2 = 16(1 + 14m^2 + m^4)\beta^2 k^4 c_3^2, \tag{30}$$

of the Boussinesq equation by

$$(c_3^2 - c_0^2)^2 = 16(1 + 14m^2 + m^4)\alpha^2 k^4, \tag{31}$$

and for the nonlinear Klein-Gordon equation (**A**) one finds

$$\alpha^2 = 16(1 + 14m^2 + m^4)k^4(c_3^2 - c_0^2)^2. \tag{32}$$

Similarly, differentiating (27) twice with respect to ξ and using (22) and (24), we have

$$e_1 \frac{d^2 u_3}{d\xi^2} + e_2 u_3 + e_3 u_3^2 + \frac{e_2^2 - 16(1 + 14m^2 + m^4)e_1^2}{4e_3} = 0. \tag{33}$$

It is obvious that, if and only if (28) is satisfied, then (27) is another travelling wave solution to (6) with the wave speed c_3.

When $m \to 1$, the combined travelling wave solution (27) reduces to

$$u_3 = a_0 + a_2 \tanh^2 \xi + a_2 \coth^2 \xi, \quad \text{with } a_0 = -\frac{e_2}{2e_3} + \frac{4e_1}{e_3}, \quad a_2 = -\frac{6e_1}{e_3}. \tag{34}$$

3. The Second Kind of Nonlinear Evolution Equations

3.1. *The Travelling Wave Solution*

The second kind of nonlinear evolution equations mKdV equation

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0, \tag{35}$$

the mBBM equation

$$u_t + c_0 u_x + \alpha u^2 u_x + \beta u_{xxt} = 0, \tag{36}$$

and the nonlinear Klein-Gordon equation (**B**)

$$u_{tt} - c_0^2 u_{xx} + \alpha u - \beta u^3 = 0. \tag{37}$$

Substituting (5) into (35) to (37), or integrating once or twice and taking the integration constants as zero, we have

$$e_1 \frac{d^2 u}{d\xi^2} + e_2 u + e_3 u^3 = 0. \tag{38}$$

Here e_1, e_2 and e_3 are again constants, which depend on (35) to (37). In case of the mKdV equation, one calculates

$$e_1 = \beta k^2, \quad e_2 = -c, \quad e_3 = \frac{\alpha}{3}, \tag{39}$$

for the mBBM equation, it is

$$e_1 = \beta k^2 c, \quad e_2 = c - c_0, \quad e_3 = -\frac{\alpha}{3}, \tag{40}$$

and for the nonlinear Klein-Gordon equation (**B**)

$$e_1 = k^2(c^2 - c_0^2), \quad e_2 = \alpha, \quad e_3 = -\beta. \tag{41}$$

3.2. The Basic Travelling Wave Solution

By using the Jacobi elliptic function expansion method [11, 12], $u(\xi)$ in (38) can be expressed for these nonlinear equations as

$$u_1 = a_1 \mathrm{sn}\xi \tag{42}$$

with

$$a_1 = \pm\sqrt{-\frac{2e_1}{e_3}}m \tag{43}$$

and the modulus $m(0 < m < 1)$ satisfying

$$e_2 = (1+m^2)e_1. \tag{44}$$

The constraint (44) can be applied to determine the wave speed of the travelling solution u_1. For the mKdV equation one calculates

$$c_1 = -(1+m^2)\beta k^2, \tag{45}$$

for the mBBM equation

$$c_1 - c_0 = 16(1+m^2)\beta k^2 c_1, \tag{46}$$

and for the nonlinear Klein-Gordon equation (**B**)

$$\alpha = (1+m^2)k^2(c_1^2 - c_0^2). \tag{47}$$

When $m \to 1$, the basic travelling wave solution (42) reduces to

$$u_1 = a_1 \tanh\xi \quad \text{with} \quad a_1 = \pm\sqrt{-\frac{2e_1}{e_3}}. \tag{48}$$

3.3. The Sister Travelling Wave Solution

Also in these cases we can show that there exists a sister travelling wave solution to (38). It has the form

$$u_2 = \frac{a_1}{m}\mathrm{ns}\xi, \quad \text{with} \quad \mathrm{ns}\xi \equiv \frac{1}{\mathrm{sn}\xi}, \tag{49}$$

where a_1 is the same constant as in (43), and also (44) holds for this sister solution (49). Denoting again the wave speed of the travelling wave solution (49) as c_2, we have

$$c_2 = c_1. \tag{50}$$

The derivations of (49) and (50) are similar to those in the previous case. We can easily prove that u_1 in (42) satisfies

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\xi^2} = -(1+m^2)u_1 + \frac{2m^2}{a_1^2}u_1^3. \tag{51}$$

In the same way it can be proven that u_2 satisfies

$$\frac{\mathrm{d}^2 u_2}{\mathrm{d}\xi^2} = -(1+m^2)u_2 + \frac{2m^2}{a_1^2}u_2^3. \tag{52}$$

Obviously, both u_2 and u_1 satisfy the same equation (38). Therefore u_2 in (49) is another travelling wave solution to (38).

When $m \to 1$, the sister travelling wave solution (49) reduces to

$$u_2 = a_1 \coth\xi, \quad \text{with} \quad a_1 = \pm\sqrt{-\frac{2e_1}{e_3}}. \tag{53}$$

3.4. Combinability of Travelling Wave Solutions

Also for these nonlinear equations we can show that the combined solution

$$u_3 = u_1 + u_2 = a_1\mathrm{sn}\xi + \frac{a_1}{m}\mathrm{ns}\xi \tag{54}$$

is also a travelling wave solution of (38). It satisfies the constraint

$$e_2 = (1+6m+m^2)e_1, \tag{55}$$

which implies that the wave speed, c_3 of (54) is different from those of (42) and (49). For the three nonlinear equation we obtain the mKdV equation

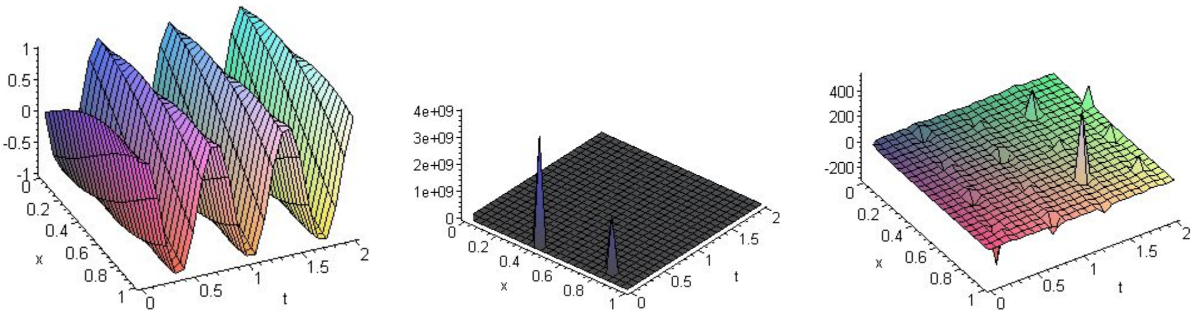
$$c_3 = -(1+6m+m^2)\beta k^2, \tag{56}$$

the mBBM equation

$$(c_3 - c_0) = (1+6m+m^2)\beta k^2 c_3, \tag{57}$$

and the nonlinear Klein-Gordon equation (**B**)

$$\alpha = (1+6m+m^2)k^2(c_3^2 - c_0^2). \tag{58}$$



(a) Basic traveling solution u_1 (b) Sister traveling solution u_2 (c) Combined traveling solution u_3

Fig. 1 Graphical presentation for various solutions of the mKdV equation when $\alpha = 6.0$, $\beta = -1.0$, $k = 2.0$ and $m = 0.5$.

Similarly, differentiating (54) twice with respect to ξ and using (51) and (52), we have

$$e_1 \frac{\mathrm{d}^2 u_3}{\mathrm{d}\xi^2} + (1 + 6m + m^2) e_1 u_3 + e_3 u_3^3 = 0. \quad (59)$$

It is obvious that, if and only if (55) is satisfied, then (54) is another travelling wave solution to (38) with wave speed c_3.

Taking the solutions of the mKdV u_1, u_2 and u_3 as an example, the graphical presentations of u_1, u_2 and u_3 for different times t and locations x are shown in Figure 1. It is obvious that the graphical presentations of u_1, u_2 and u_3 take different shapes, and the velocity c_2, which is the same as c_1, is also different from c_3.

When $m \to 1$, the combined travelling wave solution (54) reduces to

$$u_3 = a_1 \tanh\xi + a_1 \coth\xi, \text{ with } a_1 = \pm\sqrt{-\frac{2e_1}{e_3}}. \quad (60)$$

4. Conclusion

In this paper, we analyzed the combinability of travelling wave solutions to nonlinear evolution equations. It is shown that a proper linear combination of the basic and its sister travelling wave solution is also a travelling wave solutions to the same nonlinear evolution equations. However, it has a different wave speed. Generally speaking, the different solutions of nonlinear evolution equations do not allow a superposition principle, but we hope that this problem is pending for further discussion.

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